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A continuum theory of thermoelastic solids with diatomic structure

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SUMMARY

Generalizing the idea of the deformable elastic shell model by Dick and Overhauser [1], a continuum model for thermoelastic diatomic elastic solids is presented. The model used is based on the assumption that a diatomic solid may be considered to consist of two simple and initially overlapping elastic media interacting with each other. Based on this assumption, the kinematics, balance laws and the appropriate constitutive relations for heat conducting diatomic elastic solids with multiple temperature are presented. For the illustration of the theory, the propagation of time-harmonic thermal waves in elastically rigid heat-conducting diatomic solids is studied and some particular cases are discussed.

1. Introduction

It is a well-known fact that most of the elastic materials are made of complex molecules rather than simple atoms. From the viewpoint of lattice dynamics it is, therefore, apparent that the internal structure of such solids is multi-atomic. The classical continuum theory of elastic solids ignores the relative motions of particles in the same cell, and comes up with the result that the wave propagation in such an elastic medium is not dispersive. However, the results of phonon dispersion experiments (c.f. Brockhause [2], Harrison [3] and Wallis [4]) show that the phase velocity changes with wave number. These facts have forced the researchers to introduce generalized continuum theories that take the relative motion of particles into account. Among these studies it is worthy to mention the director theory of Toupin [5], the micromorphic theory of Eringen and Suhubi [6], and the multipolar theory of Green and Rivlin [7]. These theories are mathematically complete, yet they have found little applications in physical problems concerning elastic solids.

The continuum theory of elastic solids with diatomic structure was first laid down by Demiray [8, 10], in which the body is assumed to consist of two simple and initially overlapping elastic media interacting with each other. The balance laws and thermodynamically admissible constitutive equations, and related kernel functions characterizing the elastic properties of constituents are also reported in the same work. In the present work, a continuum formulation of heat-conducting elastic diatomic solids with multiple temperature is presented. The balance laws are formulated for each species in the medium whereas the entropy inequality is formulated for the whole of the body. A set of nonlinear and linear constitutive equations is derived

and several particular cases are discussed. As an application of the present derivation, the propagation of harmonic waves in elastically rigid heat conductors is studied. Contrary to the result of classical thermoelasticity, it is shown that the speed of propagation in rigid conductors is finite.

2. Kinematics and balance laws

Consider an elastic diatomic continuum whose coordinates of material particles in the undeformed body B_0 are denoted by $\mathbf{X}$. Thus the motion of each species in the body is described by

$$\mathbf{x}_{(\alpha)} = \mathbf{x}_{(\alpha)}(\mathbf{X}, t) \quad , \quad (\alpha = 1, 2). \tag{2.1}$$

The velocity $\mathbf{v}_{(\alpha)}$ and the acceleration $\mathbf{a}_{(\alpha)}$ of the αth component are defined by

$$\mathbf{v}_{(\alpha)} \equiv \left. \frac{\partial \mathbf{x}_{(\alpha)}}{\partial t} \right|_{\mathbf{X}} \equiv \dot{\mathbf{x}}_{(\alpha)} \quad , \quad \mathbf{a}_{(\alpha)} \equiv \left. \frac{\partial \mathbf{v}_{(\alpha)}}{\partial t} \right|_{\mathbf{X}} = \dot{\mathbf{v}}_{(\alpha)}. \tag{2.2}$$

For our future purposes, we define the deformation gradient, its inverse and their time derivatives as

$$\mathbf{F}_{(\alpha)} \equiv \partial \mathbf{x}_{(\alpha)} / \partial \mathbf{X} \quad , \quad \mathbf{H}_{(\alpha)} \equiv \partial \mathbf{X} / \partial \mathbf{x}_{(\alpha)} \quad , \quad \mathbf{F}_{(\alpha)} \mathbf{H}_{(\alpha)} = \mathbf{H}_{(\alpha)} \mathbf{F}_{(\alpha)} = \mathbf{I}; \tag{2.3}$$

$$\dot{\mathbf{F}}_{(\alpha)} \equiv \mathbf{L}_{(\alpha)} \mathbf{F}_{(\alpha)} \quad , \quad \dot{\mathbf{H}}_{(\alpha)} = -\mathbf{H}_{(\alpha)} \mathbf{L}_{(\alpha)} \quad , \tag{2.4}$$

where $\mathbf{L}_{(\alpha)}$ denotes the space gradient of the velocity vector $\mathbf{v}_{(\alpha)}$ of the αth component. For other details of the kinematics of diatomic solids the reader is referred to [8, 10].

Since the constituents of diatomic solids initially occupy the same material volume in space, it might be convenient to work with quantities described on the undeformed body. The balance laws are listed below:
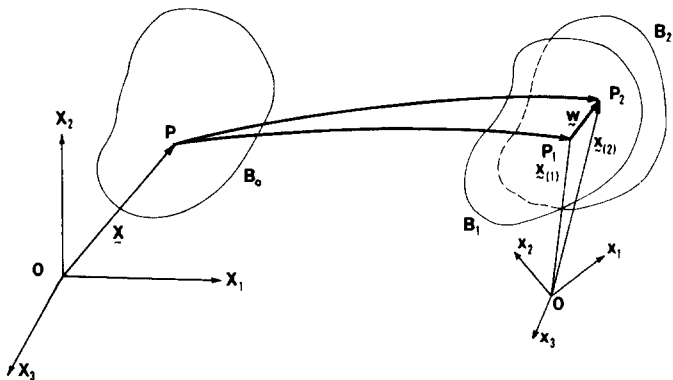

Fig. 1. Finite deformation of a diatomic solid.

(i) *Conservation of mass:*

$$\rho^0_{(\alpha)} = J_{(\alpha)} \rho_{(\alpha)} \quad , \quad (\alpha = 1, 2) \tag{2.5}$$

where $\rho^0_{(\alpha)}$, $\rho_{(\alpha)}$ and $J_{(\alpha)}$ are respectively the initial and present values of mass density and the Jacobian of the motion of the αth component of the diatomic solid.

(ii) *Balance of linear momentum:*

$$\text{div } \mathbf{T}_{(\alpha)} + \rho^0_{(\alpha)}(\mathbf{f}_{(\alpha)} - \mathbf{a}_{(\alpha)}) + \mathbf{G}_{(\alpha)} = 0 \quad , \quad (\alpha = 1, 2). \tag{2.6}$$

where $\mathbf{T}_{(\alpha)}$, $\mathbf{f}_{(\alpha)}$ and $\mathbf{G}_{(\alpha)}$ are respectively the Piola-Kirchhoff stress tensor, the body force and the rate of linear momentum transfer, subject to

$$\sum_{\alpha=1}^{2} \mathbf{G}_{(\alpha)} = 0. \tag{2.7}$$

(iii) *Balance of angular momentum:*

$$\sum_{\alpha=1}^{2} (\mathbf{F}_{(\alpha)} \mathbf{T}_{(\alpha)} - \mathbf{x}_{(\alpha)} \oplus \mathbf{G}_{(\alpha)}) = \text{Symmetric.} \tag{2.8}$$

(iv) *Conservation of energy:*

$$\rho^0_{(\alpha)} \dot{\epsilon}_{(\alpha)} = tr(\mathbf{T}_{(\alpha)} \dot{\mathbf{F}}_{(\alpha)}) - \mathbf{G}_{(\alpha)} \cdot \mathbf{v}_{(\alpha)} + \text{div } \mathbf{Q}_{(\alpha)} + \rho^0_{(\alpha)} h_{(\alpha)} + E_{(\alpha)}. \tag{2.9}$$

with

$$\sum_{\alpha=1}^{2} E_{(\alpha)} = 0$$

where $\epsilon_{(\alpha)}$ is the internal energy density per unit mass, $h_{(\alpha)}$ is the heat source per unit mass, $\mathbf{Q}_{(\alpha)}$ is the heat in-flux per unit undeformed area and $E_{(\alpha)}$ is the rate of energy transfer per unit volume of the αth component.

Re-defining the energy-rate terms in a proper way, the energy equations may be expressed as

$$\rho^0_{(\alpha)} \dot{\epsilon}_{(\alpha)} = tr(\sigma_{(\alpha)} \dot{\mathbf{S}}_{(\alpha)}) - \mathbf{G}_{(\alpha)} \cdot \dot{\mathbf{w}}_{(\alpha)} + \text{div } \mathbf{Q}_{(\alpha)} + \rho^0_{(\alpha)} h_{(\alpha)} + e_{(\alpha)} \quad , \quad \sum_{\alpha} e_{(\alpha)} = 0. \tag{2.10}$$

where

$$E_{(1)} \equiv e_{(1)} + \mathbf{G}_{(1)} \cdot \mathbf{v}_{(1)} + tr(\mathbf{T}_{(1)} \dot{\mathbf{F}}_{(1)}),$$

$$E_{(2)} \equiv e_{(2)} - \mathbf{G}_{(1)} \cdot \mathbf{v}_{(1)} - tr(\mathbf{T}_{(2)} \dot{\mathbf{F}}_{(1)}),$$

$$\sigma_{(1)} \equiv \mathbf{T}_{(1)} + \mathbf{T}_{(2)} \quad , \quad \sigma_{(2)} \equiv \mathbf{T}_{(2)} \quad , \quad \mathbf{S}_{(1)} = \mathbf{F}_{(1)} \quad , \quad \mathbf{S}_{(2)} \equiv \mathbf{F}_{(2)} - \mathbf{F}_{(1)}, \tag{2.11}$$

$$\mathbf{w}_{(\alpha)} = \mathbf{x}_{(\alpha)} - \mathbf{x}_{(1)}.$$

(v) *Entropy inequality:*

The balance of entropy for a diatomic solid is expressed by

$$\gamma_{(\alpha)} \equiv \rho_{(\alpha)}^0 \dot{\eta}_{(\alpha)} - \mathrm{div} \left(\frac{Q_{(\alpha)}}{\theta_{(\alpha)}} \right) - \frac{\rho_{(\alpha)}^0 h_{(\alpha)}}{\theta_{(\alpha)}} + n_{(\alpha)} \tag{2.12}$$

with

$$\Sigma n_{(\alpha)} = 0. \tag{2.13}$$

where $\eta_{(\alpha)}$ is the entropy volume density per unit mass, $n_{(\alpha)}$ is the rate of entropy transfer into the *αth* component, $\theta_{(\alpha)} > 0$ is the absolute temperature and $\gamma_{(\alpha)}$ is the rate of partial entropy production per unit volume of the body. Following Bowen and Wiese [11], and Dunwoody and Müller [12], the second law of thermodynamics is taken to be

$$\sum_{\alpha} \gamma_{(\alpha)} \geqslant 0. \tag{2.14}$$

In mixture theories, however, some people use separate entropy inequalities for each component in the mixture (c.f. Eringen and Ingram [13], and Green and Naghdi [14]).

Eliminating $\rho_{(\alpha)}^0 h_{(\alpha)}$ between (2.10), (2.14) and utilizing the Helmholtz free energy,

$$\psi_{(\alpha)} \equiv \epsilon_{(\alpha)} - \theta_{(\alpha)} \eta_{(\alpha)},$$

the following alternate form for the entropy inequality is obtained

$$\sum_{\alpha} \left[-\frac{\rho_{(\alpha)}^0}{\theta_{(\alpha)}} (\dot{\psi}_{(\alpha)} + \eta_{(\alpha)}\dot{\theta}_{(\alpha)}) + tr \left(\frac{\sigma_{(\alpha)}\dot{S}_{(\alpha)}}{\theta_{(\alpha)}} \right) + \frac{Q_{(\alpha)} \cdot \Lambda_{(\alpha)}}{\theta_{(\alpha)}^2} \right]$$

$$- \frac{G_{(2)} \cdot \dot{w}_{(2)}}{\theta_{(2)}} + e_{(1)} \left(\frac{1}{\theta_{(1)}} - \frac{1}{\theta_{(2)}} \right) \geqslant 0, \tag{2.15}$$

where $\Lambda_{(\alpha)} \equiv \mathrm{grad}\, \theta_{(\alpha)}$.

This inequality must be valid *for all independent thermodynamical processes.*

For a detailed discussion of the kinematics and balance equations of diatomic solids the reader is referred to Demiray [8, 10].

3. Constitutive Equations

In this section we develop a set of constitutive equations for heat conducting diatomic elastic solids with multiple temperature. The independent constitutive variables are selected to be

$$S_{(\alpha)} \quad , \quad w_{(2)} \quad , \quad \theta_{(\alpha)} \quad , \quad \Lambda_{(\alpha)} \quad , \quad (\alpha = 1, 2). \tag{3.1}$$

Thus the general form of a constitutive dependent variable may be given in the following form

$$\psi_{(\alpha)} = \psi_{(\alpha)} \left(\mathbf{S}_{(\beta)}, \mathbf{w}_{(2)}, \theta_{(\beta)}, \Lambda_{(\beta)} \right) \quad , \quad (\alpha, \beta = 1, 2). \tag{3.2}$$

A similar form of constitutive functions is valid for other dependent variables. Introducing (3.2) into (2.15) we obtain

$$\sum_{\alpha} \left\{ - \left[\sum_{\beta} \left(\frac{\rho^0_{(\beta)}}{\theta_{(\beta)}} \cdot \frac{\partial \psi_{(\beta)}}{\partial \theta_{(\alpha)}} \right) + \frac{\rho^0_{(\alpha)}}{\theta_{(\alpha)}} \eta_{(\alpha)} \right] \dot{\theta}_{(\alpha)} + tr \left[\left(\frac{\sigma_{(\alpha)}}{\theta_{(\alpha)}} - \sum_{\beta} \frac{\rho^0_{(\beta)}}{\theta_{(\beta)}} \cdot \frac{\partial \psi_{(\beta)}}{\partial \mathbf{S}_{(\alpha)}} \right) \dot{\mathbf{S}}_{(\alpha)} \right] \right.$$

$$\left. - \sum_{\beta} \frac{\rho^0_{(\beta)}}{\theta_{(\beta)}} \frac{\partial \psi_{(\beta)}}{\partial \Lambda_{(\alpha)}} \cdot \dot{\Lambda}_{(\alpha)} + \frac{\mathbf{Q}_{(\alpha)} \cdot \Lambda_{(\alpha)}}{\theta^2_{(\alpha)}} \right\} \quad \left(\frac{\mathbf{G}_{(2)}}{\theta_{(2)}} + \sum_{\beta} \frac{\rho^0_{(\alpha)}}{\theta_{(\beta)}} \frac{\partial \psi_{(\beta)}}{\partial \mathbf{w}_{(2)}} \right) \cdot \dot{\mathbf{w}}_{(2)}$$

$$+ e_{(1)} \left(- \frac{1}{\theta_{(2)}} - \frac{1}{\theta_{(1)}} \right) \geqslant 0. \tag{3.3}$$

Since the inequality (3.3) is linear in $\dot{\theta}_{(\alpha)}$, $\dot{\Lambda}_{(\alpha)}$, $\dot{\mathbf{S}}_{(\alpha)}$ and $\dot{\mathbf{w}}_{(2)}$, the necessary and sufficient conditions for the inequality to be valid for all arbitrary variations of these variables are that the coefficients of these quantities must vanish, i.e.,

$$\rho^0_{(\alpha)} \eta_{(\alpha)} = - \theta_{(\alpha)} \frac{\partial}{\partial \theta_{(\dot{\alpha})}} \left(\frac{\Sigma_{(\alpha)}}{\theta_{(\alpha)}} \right) - \frac{\rho^0_{(\alpha)}}{\theta_{(\alpha)}} \psi_{(\alpha)} \quad , \quad (\alpha = 1, 2), \tag{3.4}$$

$$\sigma_{(\alpha)} = \frac{\partial \Sigma_{(\alpha)}}{\partial \mathbf{S}_{(\alpha)}} \quad , \quad \mathbf{G}_{(2)} = - \frac{\partial \Sigma_{(2)}}{\partial \mathbf{w}_{(2)}} . \tag{3.5}$$

Here the functions $\Sigma_{(\alpha)} \left(\mathbf{S}_{(\gamma)}, \mathbf{w}_{(2)}, \theta_{(\gamma)} \right)$, $(\alpha = 1, 2)$ are defined by

$$\frac{\Sigma_{(1)}}{\theta_{(1)}} = \frac{\Sigma_{(2)}}{\theta_{(2)}} = \sum_{\beta=1}^{2} \left(\frac{\rho^0_{(\beta)} \psi_{(\beta)}}{\theta_{(\beta)}} \right) , \tag{3.6}$$

and the remainder of the inequality becomes

$$\sum_{\alpha} \left[\frac{\mathbf{Q}_{(\alpha)} \cdot \Lambda_{(\alpha)}}{\theta^2_{(\alpha)}} + \frac{e_{(\alpha)}}{\theta_{(\alpha)}} \right] \geqslant 0. \tag{3.7}$$

As is seen from equations (3.4)-(3.5) the stresses and the rate of momentum transfer terms are independent of the temperature gradients, whereas the partial entropy densities may depend on the gradients of the temperature, the sum of the entropy densities being independent of the variables mentioned.

The principle of objectivity (frame indifference) implies that the free energy should have the following form

$$\psi_{(\alpha)} = \psi_{(\alpha)} \left(\mathbf{C}, \mathbf{D}, \Gamma, \theta_{(\beta)}, \Lambda_{(\beta)} \right) \tag{3.8}$$

where

$$\mathbf{C} \equiv \mathbf{S}_{(1)}^T \mathbf{S}_{(1)} \quad , \quad \mathbf{D} \equiv \mathbf{S}_{(1)}^T \mathbf{S}_{(2)} \quad , \quad \mathbf{\Gamma} \equiv \mathbf{S}_{(1)}^T \mathbf{w}_{(2)}. \tag{3.9}$$

Here the superscript $(\ )^T$ denotes the transpose of the corresponding tensor. Substituting (3.8) into (3.4)-(3.6) we have

$$\sigma_{(1)} = 2\, \frac{\partial \Sigma_{(1)}}{\partial \mathbf{C}}\, \mathbf{S}_{(1)}^T + \frac{\partial \Sigma_{(1)}}{\partial \mathbf{D}}\, \mathbf{S}_{(2)}^T + \frac{\partial \Sigma_{(1)}}{\partial \mathbf{\Gamma}}\, \mathbf{w}_{(2)},$$

$$\sigma_{(2)} = \left(\mathbf{S}_{(2)}\, \frac{\partial \Sigma_{(2)}}{\partial \mathbf{D}} \right)^T \quad , \quad \mathbf{G}_{(2)} = -\mathbf{S}_{(1)}\, \frac{\partial \Sigma_{(2)}}{\partial \mathbf{\Gamma}}\ . \tag{3.10}$$

The set of equations (3.10) gives the most general form of constitutive equations for heat conducting elastic diatomic solids. In what follows we shall present the linear constitutive equations.

4. Linear constitutive equations

In the linear constitutive equations the constitutive independent variables may be expressed as

$$2\mathbf{E} = \mathbf{C} + \mathbf{I} \cong [\text{grad } \mathbf{u} + (\text{grad } \mathbf{u})^T] \quad , \quad \mathbf{D} \cong \text{grad } \mathbf{w} \quad , \quad \mathbf{\Gamma} \cong \mathbf{w} \cong \mathbf{w}_{(2)},$$

$$\theta_{(\alpha)} = \theta_0 + T_{(\alpha)} \quad , \quad \text{with } |T_{(\alpha)}| \ll \theta_0 \quad , \quad \theta_0 > 0, \tag{4.1}$$

where $\mathbf{E}$, $\mathbf{u}$, θ_0 and $T_{(\alpha)}$ are respectively the Lagrangian strain tensor, the Cartesian components of the displacement vector of species (1), the constant equilibrium temperature and the deviation of absolute temperature from the equilibrium temperature. Furthermore, approximating $1/\theta_{(\alpha)}$ by $(1 - T_{(\alpha)}/\theta_0)/\theta_0$, and keeping only the quadratic terms in independent variables, we have from (3.4)

$$\Sigma_{(\alpha)} = \Sigma + (-1)^\alpha\, \frac{1}{\theta_0}\, (T_{(2)} - T_{(1)})\, \text{Lin}\, (\Sigma - \rho_{(\alpha)}^0 \psi_{(\alpha)}) \quad , \quad \Sigma = \sum_\alpha \rho_{(\alpha)}^0 \psi_{(\alpha)}, \tag{4.2}$$

where the symbol (Lin) is used to denote the linear part of the corresponding quantities.

For linear and isotropic materials the total free energy Σ may be expressed by

$$\Sigma = \Sigma_0 - \sum_\alpha \rho_{(\alpha)}^0 \eta_{(\alpha)}^0 T_{(\alpha)} - \tfrac{1}{2}(\alpha_5 - 2\tau_1)T_{(1)}^2 - (\alpha_6 + \tau_1 + \tau_2)T_{(1)}T_{(2)}$$

$$- \tfrac{1}{2}(\alpha_7 - 2\tau_2)T_{(2)}^2 - (\beta_1 + \beta_3 - \tau_3)T_{(1)}tr\mathbf{E} - (\beta_2 + \beta_4 + \tau_3)T_{(2)}tr\mathbf{E}$$

$$- (\beta_3 + \tau_4)T_{(1)}tr\mathbf{D} - (\beta_4 - \tau_4)T_{(2)}tr\mathbf{D} + \tfrac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3)(tr\mathbf{E})^2 +$$

$$(\mu_1 + 2\mu_2 + \mu_3)tr(\mathbf{E}^2) + \tfrac{1}{2}\lambda_3(tr\mathbf{D})^2 + (\mu_3 + \kappa)tr(\mathbf{D}\mathbf{D}^T) + (\mu_3 - \kappa)tr\mathbf{D}^2$$

$$+ (\lambda_2 + \lambda_3)tr\mathbf{E}tr\mathbf{D} + 2(\mu_2 + \mu_3)tr(\mathbf{D}\mathbf{E}) + \tfrac{1}{2}\nu\mathbf{w}\cdot\mathbf{w}. \tag{4.3}$$

Similarly the linear parts of $\rho^0_{(\alpha)}\psi_{(\alpha)}$ may be given by

$$\text{Lin} \left(\rho^0_{(1)}\psi_{(1)} \right) = \rho^0_{(1)}\psi^0_{(1)} - \theta_0(\tau_5 T_{(1)} + \tau_2 T_{(2)} + \tau_6 tr\mathbf{E} + \tau_4 tr\mathbf{D}), \tag{4.4a}$$

$$\text{Lin} \left(\rho^0_{(2)}\psi_{(2)} \right) = \rho^0_{(2)}\psi^0_{(2)} - \theta_0(\tau_1 T_{(1)} + \tau_7 T_{(2)} + \tau_3 tr\mathbf{E} + \tau_8 tr\mathbf{D}). \tag{4.4b}$$

Substitution of (4.3) and (4.4a, b) into (3.10) yields the following linearized constitutive equations:

$$\rho^0_{(1)}\eta_{(1)} = \rho^0_{(1)}\eta^0_{(1)} + (\alpha_5 - \tau_1)T_{(1)} + (\alpha_6 + \tau_2)T_{(2)} + (\beta_1 + \beta_3 - \tau_3)tr\mathbf{E}$$
$$+ (\beta_3 + \tau_4)tr\mathbf{D}, \tag{4.5a}$$

$$\rho^0_{(2)}\eta_{(2)} = \rho^0_{(2)}\eta^0_{(2)} + (\alpha_6 + \tau_1)T_{(1)} + (\alpha_7 - \tau_2)T_{(2)} + (\beta_2 + \beta_4 + \tau_3)tr\mathbf{E}$$
$$+ (\beta_3 - \tau_4)tr\mathbf{D}, \tag{4.5b}$$

$$\sigma_{(1)} = [(\lambda_1 + 2\lambda_2 + \lambda_3)tr\mathbf{E} + (\lambda_2 + \lambda_3)tr\mathbf{D} - (\beta_1 + \beta_3)T_{(1)} - (\beta_2 + \beta_4)T_{(2)}]\mathbf{I}$$
$$+ (2\mu_1 + 4\mu_2 + 2\mu_3)\mathbf{E} + (\mu_2 + \mu_3)(\mathbf{D} + \mathbf{D}^T), \tag{4.6a}$$

$$\sigma_{(2)} = [\lambda_3 tr\mathbf{D} + (\lambda_2 + \lambda_3)tr\mathbf{E} - \beta_3 T_{(1)} - \beta_4 T_{(2)}]\mathbf{I} + (\mu_3 + \kappa)\mathbf{D}^T + (\mu_3 - \kappa)\mathbf{D} + 2(\mu_2 + \mu_3)\mathbf{E}, \tag{4.6b}$$

$$\mathbf{G}_{(2)} = -\nu w, \tag{4.7}$$

where $\mathbf{I}$ is the identity matrix and $\rho^0_{(\alpha)}\eta^0_{(\alpha)}$ is the equilibrium value of the entropy density of the αth constituent. If the body is assumed to be initially stress-free, then further conditions may be put on these coefficients, i.e.,

$$\tau_3 + \tau_6 = \tau_4 + \tau_8 = 0.$$

Similarly the heat-flux vectors and the rate of energy transfer terms may be expressed as

$$\mathbf{Q}_{(1)} = K_1 \Lambda_{(1)} + K_2 \Lambda_{(2)} + \gamma_1 \mathbf{w},$$

$$\mathbf{Q}_{(2)} = K_3 \Lambda_{(1)} + K_4 \Lambda_{(2)} + \gamma_2 \mathbf{w}, \tag{4.8}$$

$$e_{(1)} = -a_1 T_{(1)} + a_2 T_{(2)} + a_3 tr\mathbf{E} + a_4 tr\mathbf{D}.$$

If (4.8) is introduced into the linearized form of the entropy inequality (3.7), an important implication deduced from the inequality will be as follows

$$\gamma_1 = \gamma_2 = 0 \quad , \quad a_3 = a_4 = 0 \quad , \quad a_2 = -a_1 = a \geqslant 0. \tag{4.9}$$

Hence, we see that the heat flux does not depend on the relative displacement **w** and the reduced form of the energy transfer term becomes

$$e_{(1)} = a(T_{(2)} - T_{(1)}).$$ (4.10)

This shows that, in the linear constitutive theory, when the temperatures of the species are the same there will be no energy exchange between the constituents. Moreover, entropy inequality (3.7) may put some restrictions on the coefficients $K_i(i = 1, 2, 4)$, but to save space we do not list them here.

Now let us return to the investigation of energy equations (2.10). Since we assumed that the initial stresses vanish, in the linear theory the energy equations become

$$\text{div } \mathbf{Q}_{(\alpha)} + \rho^0_{(\alpha)} h_{(\alpha)} + e_{(\alpha)} - \text{Lin } \overline{[\rho^0_{(\alpha)} \epsilon_{(\alpha)}]} \dot{} = 0.$$ (4.11)

Using the relation $\psi_{(\alpha)} = \epsilon_{(\alpha)} - \theta_{(\alpha)} \eta_{(\alpha)}$ and the expressions (4.2) and (4.8), we have

$$\text{Lin } [\rho^0_{(\alpha)} \epsilon_{(\alpha)}] = \text{Lin } [\rho^0_{(\alpha)} \psi_{(\alpha)} + \rho^0_{(\alpha)} \eta_{(\alpha)} \theta_0] + \rho^0_{(\alpha)} \eta^0_{(\alpha)} T_{(\alpha)}.$$ (4.12)

Substituting (4.2), (4.5) and (4.8) into (4.11), the linearized expressions of the energy equation become

$$K_1 \nabla^2 T_{(1)} + K_2 \nabla^2 T_{(2)} + a(T_{(2)} - T_{(1)}) - \theta_0 [\alpha_5 \dot{T}_{(1)} + \alpha_6 \dot{T}_{(2)} + (\beta_1 + \beta_2) tr \dot{\mathbf{E}}$$

$$+ \beta_2 tr \dot{\mathbf{D}} + \rho^0_{(1)} h_{(1)} = 0,$$ (4.13a)

$$K_3 \nabla^2 T_{(1)} + K_4 \nabla^2 T_{(2)} + a(T_{(1)} - T_{(2)}) - \theta_0 [\alpha_6 \dot{T}_{(1)} + \alpha_7 \dot{T}_{(2)} + (\beta_2 + \beta_4) tr \dot{\mathbf{E}}$$

$$+ \beta_4 tr \dot{\mathbf{D}} + \rho^0_{(2)} h_{(2)} = 0,$$ (4.13b)

where ∇^2 is the Laplacian in three-dimensional E-space.

Noting the relation between the vectors (**u**, **w**) and $\mathbf{u}_{(\alpha)}$ ($\alpha = 1, 2$) and the tensors $\mathbf{T}_{(\alpha)}$ and $\sigma_{(\alpha)}$, e.g.,

$$\mathbf{u}_{(1)} \equiv \mathbf{u} \quad , \quad \mathbf{u}_{(2)} \equiv \mathbf{u} + \mathbf{w} \quad , \quad \mathbf{T}_{(1)} \equiv \sigma_{(1)} - \sigma_{(2)} \quad , \quad \mathbf{T}_{(2)} \equiv \sigma_{(2)},$$ (4.14)

from (4.6) and (4.7) one can express the stress tensors $T_{(\alpha)}$ in terms of the gradients of $\mathbf{u}_{(\alpha)}$. Since it is a straightforward substitution we will not repeat hem them here.

Field Equations:

Having these linearized constitutive relations we may obtain the field equations governing heat conducting elastic diatomic solids. Introducing (4.6) and (4.7) into (2.6) and noting the relations (4.14) the following differential equations are obtained:

$$(\lambda_1 + \mu_1 - \kappa) \nabla \nabla \cdot \mathbf{u}_{(1)} + (\mu_1 + \kappa) \nabla^2 \mathbf{u}_{(1)} + (\lambda_2 + \mu_2 + \kappa) \nabla \nabla \cdot \mathbf{u}_{(2)} + (\mu_2 - \kappa) \nabla^2 \mathbf{u}_{(2)}$$

$$+ \nu(\mathbf{u}_{(2)} - \mathbf{u}_{(1)}) - \beta_1 \nabla T_{(1)} - \beta_2 \nabla T_{(2)} + \rho_{(1)}^0(\mathbf{f}_{(1)} - \ddot{\mathbf{u}}_{(1)}) = 0, \qquad (4.15)$$

$$(\lambda_2 + \mu_2 + \kappa)\nabla\nabla \cdot \mathbf{u}_{(1)} + (\mu_2 - \kappa)\nabla^2 \mathbf{u}_{(1)} + (\lambda_3 + \mu_3 - \kappa)\nabla\nabla \cdot \mathbf{u}_{(2)} + (\mu_3 + \kappa)\nabla^2 \mathbf{u}_{(2)}$$

$$- \nu(\mathbf{u}_{(2)} - \mathbf{u}_{(1)}) - \beta_3 \nabla T_{(1)} - \beta_4 \nabla T_{(2)} + \rho_{(2)}^0(\mathbf{f}_{(2)} - \ddot{\mathbf{u}}_{(2)}) = 0. \qquad (4.16)$$

These field equations (4.13)-(4.16) may be used in solving some properly posed initial and/or boundary value problems of heat conducting elastic diatomic solids.

5. Wave propagation in elastically rigid heat conducting diatomic solids

As is known from the classical theory of thermoelasticity, there is no wave propagation in elastically rigid heat conducting monoatomic solids. In other words, any thermal disturbance created at any material point of rigid conductors is instantaneously felt by other particles at far distances. This means that in such solids the thermal disturbance propagates with an infinite speed. This result seems to be reasonable for the case of high temperature, where the vibrating particles convey certain information to the next particles with almost an infinite speed. However, for the case of low temperature physics the speed of information carrying particles is finite; consequently, the speed of thermal disturbances must be finite. Such a result contradicts the findings by the classical theory of thermoelasticity of monoatomic solids. In order to bridge the gap between the existing theories and the physical observations, a refined mechanical model must be presented for the thermal behavior of elastic solids. Among such efforts we may mention the works by Müller [15], Green and Laws [16], and Suhubi [17], where they have treated the entropy flux to be independent of heat flux and introduced the time rate of temperature as an independent constitutive variable. Since in such cases the final energy equation includes the second-order time derivative, as well as second-order space derivatives of temperature, the solution admits a wave solution with a finite speed.

In the present work we will show the existence of thermal waves in elastically rigid heat conducting solids by use of a different mechanical model. The theoretical model that we would like to use here is the heat conducting elastic diatomic solid with multiple temperature distribution. The field equations of elastically rigid heat conductors may be obtained from equations (4.13)-(4.16) by setting $\mathbf{E}_{(\alpha)} = 0 \ (\alpha = 1, 2)$. If this is done the result follows

$$K_1 \nabla^2 T_{(1)} + K_2 \nabla^2 T_{(2)} + a(T_{(2)} - T_{(1)}) - \theta_0(\alpha_5 \dot{T}_{(1)} + \alpha_6 \dot{T}_{(2)}) + \rho_{(1)}^0 h_{(1)} = 0, \qquad (5.1)$$

$$K_3 \nabla^2 T_{(1)} + K_4 \nabla^2 T_{(2)} + a(T_{(1)} - T_{(2)}) - \theta_0(\alpha_6 \dot{T}_{(1)} + \alpha_7 \dot{T}_{(2)}) + \rho_{(2)}^0 h_{(2)} = 0. \qquad (5.2)$$

Here we consider the propagation of simple harmonic waves in the absence of heat-source terms. The appropriate form of such a wave, propagating in the $x_1 = x$ direction, is given by

$$T_{(\alpha)} = \theta_0 T_{(\alpha)}^0 \exp[i(kx - \omega t)], \quad (\alpha = 1, 2) \qquad (5.3)$$

where $T_{(\alpha)}^0$ is the complex amplitude of the wave, k is the wave number and ω is the angular

frequency of the wave. Substituting (5.3) into (5.1) and (5.2) the following homogeneous equations are obtained

$$(i\alpha_5\theta_0\omega - K_1 k^2 - a)T_{(1)}^0 + (i\alpha_6\theta_0\omega - K_2 k^2 + a)T_{(2)}^0 = 0, \tag{5.4}$$

$$(i\alpha_6\theta_0\omega - K_3 k^2 + a)T_{(1)}^0 + (i\alpha_7\theta_0\omega - K_4 k^2 - a)T_{(2)}^0 = 0. \tag{5.5}$$

In order to have a non-zero solution for $T_{(1)}^0$ and $T_{(2)}^0$, the determinant of the coefficient matrix obtained from equations (5.4) and (5.5) must vanish, i.e.,

$$c_0\omega^2 + i(c_1 + c_2 k^2)\omega - (c_3 + c_4 k^2)k^2 = 0, \tag{5.6}$$

where, for brevity, we have set

$$c_0 = (\alpha_5\alpha_7 - \alpha_6^2)\theta_0^2 \quad , \quad c_1 = a\theta_0(\alpha_5 + \alpha_7 + 2\alpha_6) \quad , \quad c_2 = \theta_0[\alpha_5 K_4 + \alpha_7 K_1 - \alpha_6(K_2 + K_3)],$$

$$c_3 = a(K_1 + K_2 + K_3 + K_4) \quad , \quad c_4 = K_1 K_4 - K_2 K_3.$$

Assuming that k is a real quantity, according to representation (5.3), in order to have a physically admissible wave solution one must have $Im(\omega) \leqslant 0$. Setting $k = 0$ in equation (5.6) the cut-off frequencies are obtained as

$$\omega_{0,1} = 0 \quad , \quad \omega_{0,2} = -ic/c_0 \quad , \quad \text{provided} \quad c_0 \neq 0. \tag{5.7}$$

For the reasons mentioned above c_0 and c_1 must satisfy the condition $(c_1/c_0) > 0$.

Replacing ω in (5.6) by $i\Omega$, an alternate form for the dispersion equation may be obtained:

$$c_0\Omega^2 + (c_1 + c_2 k^2)\Omega + (c_3 + c_4 k^2)k^2 = 0. \tag{5.8}$$

The roots of this equation are given by

$$\Omega_{1,2} = \frac{1}{2c_0}[-(c_1 + c_2 k^2) \pm i\Delta^{1/2}(k)] \tag{5.9}$$

where

$$\Delta(k) \equiv 4c_0(c_3 + c_4 k^2)k^2 - (c_1 + c_2 k^2)^2. \tag{5.10}$$

In order to have an admissible wave solution one must have $\Delta > 0$ for some values of k. It is seen that for small wave numbers, i.e., $k \cong 0$, this condition is not satisfied. However for sufficiently large values of k, $\Delta(k)$ may have positive values, that is, thermal waves may propagate in rigid heat conductors.

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